GCSE Maths – Algebra

Completing the Square (Higher Only)

Worksheet

WORKED SOLUTIONS

This worksheet will show you how to work out different types of completing the square questions. Each section contains a worked example, a question with hints and then questions for you to work through on your own.

Section A

Worked Example

Complete the square for the quadratic expression $3x^2 + 6x - 10$

Step 1: If necessary, factorise the equation by the coefficient of x^2.
This step can be skipped if the coefficient is 1.

For this equation we factorise out the 3*:*

$$3x^2 + 6x - 10 = 3\left(x^2 + 2x - \frac{10}{3}\right)$$

Step 2: Complete the square of the part of the expression which is in the form $x^2 + bx + c$.
Start by identifying the coefficients b and c.

$$x^2 + bx + c = x^2 + 2x - \frac{10}{3}$$

$$b = 2, \qquad c = -\frac{10}{3}$$

Step 3: Divide the b coefficient by 2. This new value will be the value added to the x in the square part of the completed expression, i.e. $\left(x + \frac{b}{2}\right)^2$.

$$\frac{2}{2} = 1$$

Step 4: Substitute in the values to the completed square form.

General formula: $\boldsymbol{x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c}$

$$x^2 + 2x - \frac{10}{3} = (x + 1)^2 - 1^2 - \frac{10}{3}$$

$$3\left(x^2 + 2x - \tfrac{10}{3}\right) = 3\left[(x + 1)^2 - 1^2 - \tfrac{10}{3}\right]$$

Step 5: Simplify the expression.

$$3\left[(x + 1)^2 - 1^2 - \tfrac{10}{3}\right] = 3\left[(x + 1)^2 - \tfrac{13}{3}\right] = 3(x + 1)^2 - 13$$

Hence, $\boldsymbol{3x^2 + 6x - 10 = 3(x + 1)^2 - 13.}$

Guided Example

Complete the square for the quadratic expression $x^2 + 12x + 9$

Step 1: If necessary, factorise the equation by the coefficient of x^2.
This step can be skipped if the coefficient is 1.

$x^2 + 12x + 9$ → coefficient = 1

Step 2: Complete the square of the part of the expression which is in the form $x^2 + bx + c$.
Start by identifying the coefficients b and c.

$x^2 + 12x + 9 = x^2 + bx + c$

$b = 12 \quad c = 9$

Step 3: Divide the b coefficient by 2. This new value will be the value added to the x in the square part of the completed expression, i.e. $\left(x + \frac{b}{2}\right)^2$.

$\left(x + \frac{12}{2}\right)^2 = (x+6)^2$

Step 4: Substitute in the values to the completed square form.

General formula: $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

$x^2 + 12x + 9 = (x+6)^2 - (6)^2 + 9$

$= (x+6)^2 - 36 + 9$

$= (x+6)^2 - 27$

Step 5: Simplify the expression.

$x^2 + 12x + 9 = (x+6)^2 - 27$

Now it's your turn!
If you get stuck, look back at the worked and guided examples.

1. Complete the square for the following quadratic expressions:

a) $x^2 + 6x + 6$

$b = 6 \quad c = 6$

$x^2 + 6x + 6 = (x+3)^2 - 3^2 + 6$

$= (x+3)^2 - 9 + 6$

$= (x+3)^2 - 3$

General Formula

$x^2 + bx + c = (x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c$

b) $2x^2 + 12x + 18$

Factorise out 2:

$2[x^2 + 6x + 9]$

$x^2 + 6x + 9 \quad : \quad b = 6 \quad c = 9$

$x^2 + 6x + 9 = (x+3)^2 - 3^2 + 9$

$= (x+3)^2 - 9 + 9$

$= (x+3)^2$

$2(x^2 + 6x + 9) = 2(x+3)^2$

c) $3x^2 + 24x - 5$

Factorise out 3:

$3\left[x^2 + 8x - \frac{5}{3}\right]$

$x^2 + 8x - \frac{5}{3}$: $b = 8$ $c = -\frac{5}{3}$

$x^2 + 8x - \frac{5}{3} = (x+4)^2 - 4^2 - \frac{5}{3}$

$= (x+4)^2 - 16 - \frac{5}{3}$

$= (x+4)^2 - \frac{53}{3}$

$3\left(x^2 + 8x - \frac{5}{3}\right) = 3\left[(x+4)^2 - \frac{53}{3}\right]$

$= 3(x+4)^2 - 53$

d) $-2x^2 + 14x - 7$

Factorise out -2:

$-2\left[x^2 - 7x + \frac{7}{2}\right]$

$x^2 - 7x + \frac{7}{2}$: $b = -7$ $c = \frac{7}{2}$

$= (x - \frac{7}{2})^2 - (-\frac{7}{2})^2 + \frac{7}{2}$

$= (x - \frac{7}{2})^2 - \frac{49}{4} + \frac{7}{2}$

$= (x - \frac{7}{2})^2 - \frac{35}{4}$

$-2\left(x^2 - 7x + \frac{7}{2}\right) = -2\left[(x - \frac{7}{2})^2 - \frac{35}{4}\right]$

$= -2(x - \frac{7}{2})^2 + \frac{35}{2}$

Section B

Worked Example

Calculate the turning point and line of symmetry of $3x^2 + 6x - 10 = 0$

Step 1: Complete the square for this quadratic equation.

$$3(x + 1)^2 - 13$$

Step 2: For a quadratic equation in the form $a(x + p)^2 + q$, the turning point occurs at $(-p, q)$. Identify values for p and q and hence identify the turning point.

Here,

$$a(x + p)^2 + q = 3(x + 1)^2 - 13$$

$$p = 1, \quad q = -13$$

The turning point is $(-p, q)$
Therefore, the turning point of this equation is: $(-1, -13)$

Step 3: The line of symmetry occurs at $x = -p$. Identify the line of symmetry for this quadratic.

The line of symmetry for this equation is at $x = -1$.

Guided Example

Calculate the turning point and line of symmetry of $x^2 + 12x + 9 = 0$

Step 1: Complete the square for this quadratic equation.

$x^2+12x+9 = (x+6)^2 - (6)^2 + 9$

$= (x+6)^2 - 27$

Step 2: Identify values for p and q and hence identify the turning point.

$p = 6 \quad q = -27$

$(-p, q) = (-6, -27)$

For a quadratic equation of the form $a(x+p)^2 + q$
Turning Point: $(-p, q)$

Step 3: The line of symmetry occurs at $x = -p$. Identify the line of symmetry for this quadratic.

$x = -6$

Now it's your turn!
If you get stuck, look back at the worked and guided examples.

2. Find the turning point and line of symmetry of each of the following quadratic equations:

For the quadratic equation of the form
$a(x+p)^2+q$
Turning Point = $(-p, q)$

a) $x^2 + 4x + 9 = 0$

Complete the square:

$x^2+4x+9 = (x+2)^2 - 2^2 + 9$

$= (x+2)^2 + 5$

$p = 2 \quad q = 5$

Turning Point: $(-p, q) = (-2, 5)$

Line of Symmetry: $x = -p \Rightarrow x = -2$

b) $4x^2 - 20x + 1 = 0$

Complete the square:

$4x^2 - 20x + 1 = 4(x^2 - 5x + \frac{1}{4})$

$= 4((x^2 - \frac{5}{2})^2 - (-\frac{5}{2})^2 + \frac{1}{4})$

$= 4((x^2 - \frac{5}{2})^2 - \frac{25}{4} + \frac{1}{4})$

$= 4(x^2 - \frac{5}{2})^2 - 24$

$p = -\frac{5}{2} \quad q = -24$

Turning Point: $(-p, q) = (\frac{5}{2}, -24)$

Line of Symmetry: $x = -p \Rightarrow x = \frac{5}{2}$

c) $6x^2 + 3x - 9 = 0$

Complete the square:

$6x^2+3x-9 = 6\left(x^2+\frac{x}{2}-\frac{3}{2}\right)$

$= 6\left((x+\frac{1}{4})^2-(\frac{1}{4})^2-\frac{3}{2}\right)$

$= 6\left((x+\frac{1}{4})^2-\frac{25}{16}\right)$

$= 6(x+\frac{1}{4})^2 - \frac{75}{8}$

$p = \frac{1}{4}$ $q = \frac{-75}{8}$

Turning Point: $(-p, q) = (-\frac{1}{4}, -\frac{75}{8})$

Line of Symmetry: $x = -p \Rightarrow x = -\frac{1}{4}$

d) $-4x^2 + 12x - 1 = 0$

Complete the square:

$-4x^2+12x-1 = -4\left(x^2-3x+\frac{1}{4}\right)$

$= -4\left((x-\frac{3}{2})^2-(-\frac{3}{2})^2+\frac{1}{4}\right)$

$= -4\left((x-\frac{3}{2})^2-2\right)$

$= -4(x-\frac{3}{2})^2+8$

$p = -\frac{3}{2}$ $q = 8$

Turning Point: $(-p, q) = (\frac{3}{2}, 8)$

Line of Symmetry: $x = -p \Rightarrow x = \frac{3}{2}$

Section C

Worked Example

By completing the square, solve the quadratic equation $3x^2 + 6x - 10 = 0$

Step 1: Complete the square.

$$3x^2 + 6x - 10 = 3(x+1)^2 - 13 = 0$$

Step 2: Solve the equation by rearranging to make x the subject.

$$3(x+1)^2 - 13 = 0$$

$$3(x+1)^2 = 13$$

$$(x+1)^2 = \frac{13}{3}$$

$$x + 1 = \pm\sqrt{\frac{13}{3}}$$

$$x = -1 \pm \sqrt{\frac{13}{3}}$$

The solutions to this quadratic equation are

$$x = -1 + \sqrt{\frac{13}{3}}, \qquad x = -1 - \sqrt{\frac{13}{3}}$$

Guided Example

By completing the square, solve the quadratic equation $x^2 + 14x + 9 = 0$

Step 1: Complete the square.

$x^2+14x+9 = (x+7)^2 - (7)^2 + 9 = (x+7)^2 - 40$

Step 2: Solve the equation by rearranging to make x the subject.

$(x+7)^2 - 40 = 0$

$(x+7)^2 = 40$

$(x+7) = \pm\sqrt{40}$

$x = -7 \pm \sqrt{40}$

$x = -7+\sqrt{40}$ or $x = -7-\sqrt{40}$

Now it's your turn!
If you get stuck, look back at the worked and guided examples.

3. By completing the square, solve the following quadratic equations:

a) $x^2 + 6x + 3 = 0$

$(x+3)^2 - 9 + 3 = 0$

$(x+3)^2 - 6 = 0$

$(x+3)^2 = 6$

$(x+3) = \pm\sqrt{6}$

$x = -3 \pm \sqrt{6}$

$x = -3+\sqrt{6}$ or $x = -3-\sqrt{6}$

b) $2x^2 - 4x - 10 = 0$

$2(x^2 - 2x - 5) = 0$

As we are solving equal to zero

↳ we can divide both sides by 2

$x^2 - 2x - 5 = 0$

$(x-1)^2 - 1^2 - 5 = 0$

$(x-1)^2 - 6 = 0$

$(x-1)^2 = 6$

$(x-1) = \pm\sqrt{6}$

$x = 1 \pm \sqrt{6}$

$x = 1+\sqrt{6}$ or $x = 1-\sqrt{6}$

c) $2x^2 + 12x + 17 = 0$

Divide both sides by 2:

$$x^2 + 6x + \frac{17}{2} = 0$$

$$(x+3)^2 - 3^2 + \frac{17}{2} = 0$$

$$(x+3)^2 - 9 + \frac{17}{2} = 0$$

$$(x+3)^2 - \frac{1}{2} = 0$$

$$(x+3)^2 = \frac{1}{2}$$

$$(x+3) = \pm\sqrt{\frac{1}{2}}$$

$$x = -3 \pm \sqrt{\frac{1}{2}}$$

$x = -3+\sqrt{\frac{1}{2}}$ or $x = -3-\sqrt{\frac{1}{2}}$

d) $3x^2 + 9x - 14 = 0$

Divide both sides by 3:

$$x^2 + 3x - \frac{14}{3} = 0$$

$$(x+\frac{3}{2})^2 - (\frac{3}{2})^2 - \frac{14}{3} = 0$$

$$(x+\frac{3}{2})^2 - \frac{9}{4} - \frac{14}{3} = 0$$

$$(x+\frac{3}{2})^2 - \frac{83}{12} = 0$$

$$(x+\frac{3}{2})^2 = \frac{83}{12}$$

$$(x+\frac{3}{2}) = \pm\sqrt{\frac{83}{12}}$$

$$x = -\frac{3}{2} \pm \sqrt{\frac{83}{12}}$$

$x = -\frac{3}{2} + \sqrt{\frac{83}{12}}$ or $x = -\frac{3}{2} - \sqrt{\frac{83}{12}}$